Optimal Choice of Thermodynamic Variables and Vertical Grid Staggering Based on Normal Modes

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1. Introduction

In the design of numerical models for simulating atmospheric flow there is ongoing debate over which choice of predicted variables is best (for example, temperature T or potential temperature θ may be chosen as one of the thermodynamic variables) and over which arrangement of the predicted variables in space is best (e.g. the Lorenz (1960) or Charney-Phillips (1953) grids in the vertical). The two issues are inextricably linked; the optimal grid will depend on the choice of variables, and vice-versa. We must therefore seek the optimal overall configuration of variables *and* grid.

Possible criteria for deciding on the optimal configuration include conservation properties and coupling between the equations for the resolved dynamics and the parameterized physics. In this report we concentrate on the ability of different candidate configurations to simulate accurately the normal modes of the linearized governing equations, particularly their dispersion properties. For example, some configurations can give wave group velocities of the wrong sign for shorter-scale waves, or have grid-scale "computational modes" that do not propagate; these are considered especially damaging to numerical solutions.

In this report we focus on the choice of thermodynamic variables, and on the choice of vertical grid staggering. For different candidate configurations the normal mode frequencies for the discrete linearized governing equations are computed and compared with the analytic normal mode frequencies.

2. Governing equations

We consider the full compressible Euler equations in planar geometry and using height z as the vertical coordinate. To begin with, the latitudinal variation of the Coriolis parameter $f = 2\Omega \sin\phi$ is neglected (we will attempt to include it later), and the $2\Omega\cos\phi$ Coriolis terms associated with the horizontal component of the Earth's rotation vector are neglected. Moist effects and diabatic heating and friction are also neglected, and the equations are linearized about a reference state (indicated by subscript s) at rest and in hydrostatic balance. In standard notation the resulting equations are

$$u_t - fv + \frac{1}{\rho_s} p_x = 0, \tag{2.1}$$

$$v_t + fu + \frac{1}{\rho_s} p_y = 0, \tag{2.2}$$

$$w_t + \frac{1}{\rho_s} p_z + \frac{g\rho}{\rho_s} = 0, \tag{2.3}$$

$$\theta_t + w\theta_{sz} = 0, \tag{2.4}$$

$$\rho_t + \nabla.(\rho_s \mathbf{u}) = \mathbf{0}, \tag{2.5}$$

$$T_t + \frac{\kappa T_s^{2-1/\kappa}}{1-\kappa} \nabla.(T_s^{1/\kappa - 1}\mathbf{u}) = \mathbf{0}, \tag{2.6}$$

$$p_t + \frac{p_s^{\kappa}}{1-\kappa} \nabla.(p_s^{1-\kappa}\mathbf{u}) = \mathbf{0}. \tag{2.7}$$

Only two of equations (2.4)-(2.7) are needed, since the evolution of the thermodynamic variables that are not predicted is implied by the linearized ideal gas law

$$p/p_s = T/T_s + \rho/\rho_s \tag{2.8}$$

and the linearized definition of potential temperature

$$\theta/\theta_s = T/T_s - \kappa p/p_s. \tag{2.9}$$

We may choose any two of the four thermodynamic variables θ, ρ, T, and p to be our predicted variables, giving 6 possible pair choices to consider. This certainly does not exhaust the possibilities. However, many of the plausible alternative choices are equivalent, or virtually equivalent, under the linear normal mode analysis to the cases being considered.

We seek solutions that satisfy the boundary conditions $w = 0$ at $z = 0$ and $z = D$. Because all coefficients in (2.1)-(2.7) are independent of x, y and t, horizontally wavelike solutions proportional to $\exp i(kx + ly - \sigma t)$ are possible. Assuming solutions of this form, (2.1)-(2.2) become

$$-i\sigma u - fv + \frac{1}{\rho_s} ikp = 0, \tag{2.10}$$

$$-i\sigma v + fu + \frac{1}{\rho_s} ilp = 0, \tag{2.11}$$

On an f-plane there is no β-effect and therefore no Rossby restoring mechanism, so all Rossby modes have zero frequency. Consequently, analysis of discrete normal modes on an f-plane can give very little useful information about how well a given scheme will represent Rossby modes. Nevertheless, Rossby modes are meteorologically important, so it is desirable to be able to assess how well candidate configurations represent them. However, if we simply replace the constant f in equations (2.1) and (2.2) by a linear latitude variation $f = f_0 + \beta y$ then solutions proportional to $\exp i(kx + ly - \sigma t)$ are no longer possible. We can attempt to include the β-effect by making an approximation analogous to that made in deriving the quasigeostrophic β-plane equations. We begin with (2.1) and (2.2) but regard f as a function of latitude, and take the horizontal divergence and vertical component of the curl of these equations. From this point regard f and β as constants, so solutions proportional to $\exp i(kx + ly - \sigma t)$ are possible. The resulting divergence and vorticity equations can be rearranged to obtain equations for u and v tendencies

$$-i\sigma u - fv + \frac{\beta}{K^2}(ilv - iku) + \frac{1}{\rho_s} ikp = 0, \tag{2.12}$$

$$-i\sigma v + fu - \frac{\beta}{K^2}(ikv + ilu) + \frac{1}{\rho_s} ilp = 0. \tag{2.13}$$

These equations are similar to (2.10) and (2.11) except for the inclusion of the β terms. An unfortunate by-product of the approximations made in deriving them is that the resulting system of equations no longer conserves energy, leading to growing and decaying normal modes, unless the meridional wavenumber $l = 0$. Nevertheless, provided we are prepared to consider only zonally propagating modes, the resulting system of equations leads to a realistic dispersion relation for acoustic, gravity, and Rossby modes. Moreover, the β terms are brought into the equations through their interaction with the horizontal velocity, as they would be in the unapproximated equations. Therefore, comparison of discrete and analytic normal modes based on equations (2.12) and (2.13) (with $l = 0$) should provide an accurate assessment of different numerical schemes in the presence of the β-effect, in particular their ability to represent Rossby modes.

3. Analytical solution

When the reference state is isothermal, implying that the reference static stability $N_s^2 = g\theta_{sz}/\theta_s$ and the reference sound speed squared $c_s^2 = RT_s/(1-\kappa)$ are both constant, the dispersion relation may be found analytically. For external modes the dispersion relation is a cubic equation for σ; the three roots correspond to two acoustic modes and a Rossby mode. For each internal mode the dispersion relation is a quintic equation for σ; the five roots correspond to two acoustic modes, two inertia-gravity modes, and a Rossby mode. The dispersion relation can easily be solved numerically for σ when the other parameters are known.

4. Discrete solution

As discussed already in Section 2, we consider 6 possible pair choices for the prognostic variables. For each of these choices there are various ways of staggering the variables. It is natural to store w at the boundaries in order to impose $w = 0$ there. Therefore, we will consider cases in which the other variables are staggered or not staggered relative to w. Moreover, it is clear from the governing equations that inaccuracies due to vertical averaging will be introduced, with no advantage gained, if u and v are not stored at the same levels. Therefore we will restrict attention to cases in which u and v are stored at the same levels. Thus, for each pair choice of prognostic variables there are 8 choices of vertical staggering, depending on whether (u, v) and the two thermodynamic variables are or are not staggered relative to w. Thus there are $6 \times 8 = 48$ cases to consider in total.

For each of these 48 cases (2.12), (2.13), (2.3), and the appropriate two equations from (2.4) to (2.7) were discretized on a grid with N full-levels and constant spacing Δz. The results shown in Section 5 are for $N = 20$. Simple centred differences over Δz or $2\Delta z$ were used to approximate vertical derivatives, and a simple average of neighbouring values was used to transfer values from half-levels to full-levels or vice-versa when needed. The discretization results in a matrix eigenvalue problem which can be solved using a standard package.

The following parameter values were used: $D = 10000\text{m}$, $g = 9.80616\text{ms}^{-2}$, $f = 1.031 \times 10^{-4}\text{s}^{-1}$, $\beta = 1.619 \times 10^{-11}\text{s}^{-1}\text{m}^{-1}$ (corresponding to 45°N), $\kappa = 0.2856$, $T_s = 250\text{K}$, $k = 2\pi \times 10^{-6}\text{m}^{-1}$, and $l = 0$.

5. Results

Many of the 48 configurations considered produce unstable modes or severely distorted numerical dispersion relations, making them unsuitable for practical use. Figure 1 compares the numerical and analytical dispersion relations for three of the more suitable configurations. The left panels are for a Charney-Phillips grid using p and θ. Diamonds indicate analytic frequencies, crosses indicate numerical frequencies. Thuburn et al. (2002) have argued on the basis of the analytical normal mode structures that this configuration should be optimal for capturing the normal modes. The present calculation confirms that it is indeed the best of all those considered at capturing all the mode frequencies. The centre panels are for a similar configuration but with ρ instead of p. This is the configuration used in the Met Office "New Dynamics". It is only slightly worse than the best configuration at capturing the Rossby mode frequencies, and might have better conservation properties. The right hand panels are for a Lorenz grid using T and ρ. Very similar results are obtained for the other Lorenz grid configurations. Lorenz grids are generally regarded as better than Charney-Phillips grids for capturing conservation properties. Again, the mode frequencies are very well captured except for the higher internal Rossby modes. However, what the figure does not show is that all of the Lorenz grid configurations have a computational mode, consisting of a two-grid oscillation in one or both of the thermodynamic variables, with zero frequency.

References

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