Development of a finite element ocean model at Imperial College with adaptive meshing

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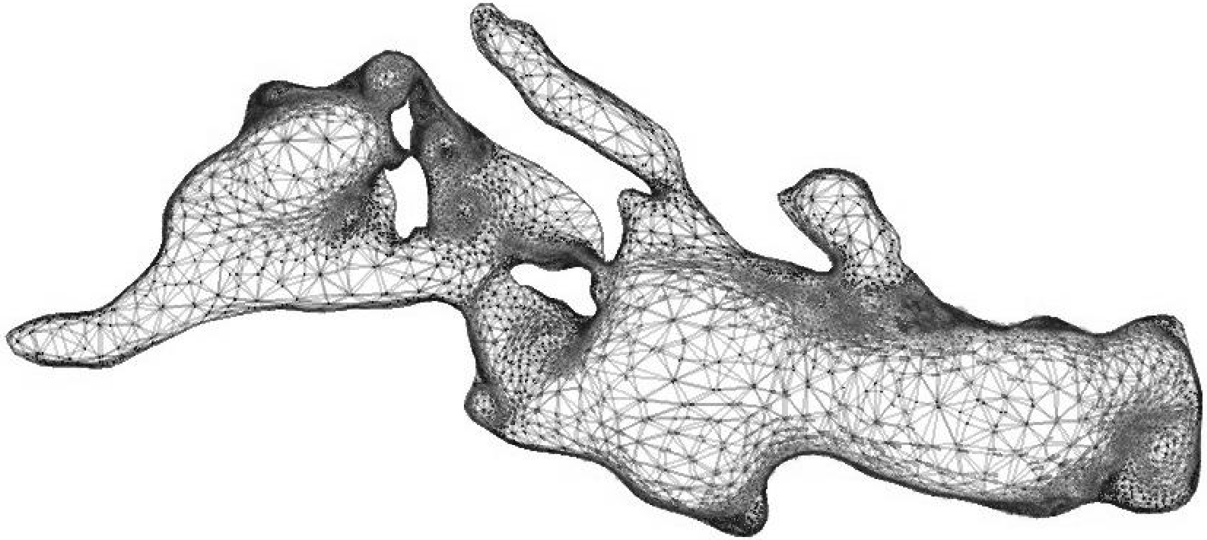


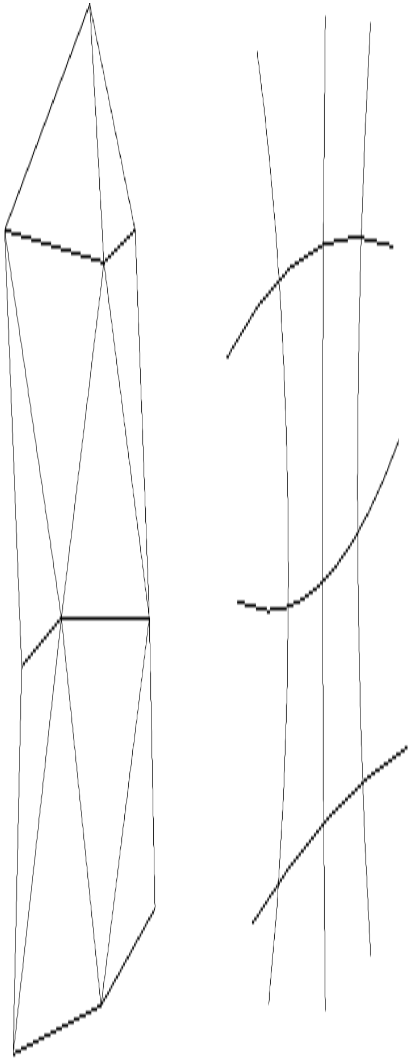
Rupert Ford (1968-2001)

Motivation

The vast majority of ocean models employ finite-difference methods and *structured grids*.

However *unstructured meshes* offer several potential advantages:

- can conform very accurately to the basin geometry;
- allow for variable resolution and mesh adaptivity;
- various ‘natural’ boundary conditions can be included in a very straightforward manner;
- formulation based on rigorous mathematical foundations, allowing statements about errors, convergence, etc …, to be made.



Application of finite-elements/unstructured meshes to the ocean

- Compared with finite-difference methods, there have been relatively few applications of finite elements/unstructured meshes to ocean models;
- A number of notable exceptions, for example: Lynch et al. (“Quoddy”), Bogden et al., Myers and Weaver, Nechaev et al., Le Roux et al., Hanert et al., Dupont et al., ….

Reasons probably include:

- Reluctance to learn the “new” technology;
- Computational overhead of finite-element method (unless combined with adaptive meshing);
- Difficulties with accurate representation of geostrophic and hydrostatic balance.

To leading order the ocean is in geostrophic and hydrostatic balance:

$$\frac{\partial u}{\partial t} + \ \dots \quad - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} \quad = \quad \dots$$

$$\frac{\partial v}{\partial t} + \ \dots \quad + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} \quad = \quad \dots$$

$$\frac{\partial w}{\partial t} + \ \dots \quad + g + \frac{1}{\rho}\frac{\partial p}{\partial z} \quad = \quad \dots$$

Small truncation errors in the representation of these balances can swamp the residual acceleration and corrupt the dynamical evolution of the flow.

To represent geostrophic and hydrostatic balance most accurately, need p to be represented by a higher-order polynomial than u, v, ρ.

However the LBB stability condition requires precisely the opposite: higher-order polynomials for u, v, ρ than for p.

Possible approaches:

- Use special element pairs that can represent geostrophic balance well (but do they really satisfy LBB?)

(Le Roux et al., 1998)

(2) Apply smoothing (and take small time-steps) to damp the computational modes

(e.g., Nechaev et al., 2003)

(3) Attempt to remove the geostrophic and hydrostatic balances from the equations of motion prior to integration

The idea:

Decompose the sum of the Coriolis and gravitational accelerations into *divergent* and *rotational* components:

$$2\Omega \times \mathbf{u} + \frac{\rho}{\rho_0}\mathbf{g} = \nabla \phi + \nabla \times \mathbf{A}$$

divergent component-
balances pressure gradient

rotational component-
(small) residual acceleration

The momentum equation becomes:

$$\frac{\partial \mathbf{u}}{\partial t} + \ldots + \nabla \times \mathbf{A} + \frac{1}{\rho_0}\nabla p_{ag} = \ldots$$

and involves only the residual ageostrophic pressure, $p_{ag} + \rho_0 \phi$
⇒should be able to use standard element pairs.

Approach is analogous to forming and solving a vorticity equation; requires solution of elliptic problem for ϕ (or $\mathbf{A}$).

Elliptic problem for **A**:

tilting of planetary vortex tubes

baroclinic production of vorticity

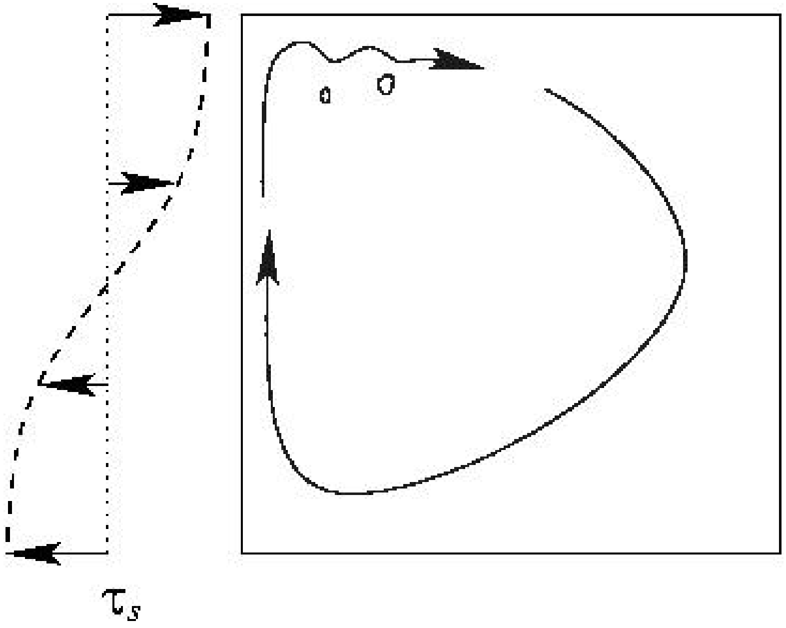
$$
\begin{aligned}
\nabla^2 A^{(x)} &= \boxed{f\,\frac{\partial u}{\partial z}} - \boxed{\frac{g}{\rho_0}\frac{\partial \rho}{\partial y}}\ ,\\
\nabla^2 A^{(y)} &= \boxed{f\,\frac{\partial v}{\partial z}} + \boxed{\frac{g}{\rho_0}\frac{\partial \rho}{\partial x}}\ ,\\
\nabla^2 A^{(z)} &= \boxed{f\,\frac{\partial w}{\partial z}} - \boxed{\beta v}\ .
\end{aligned}
$$

stretching of planetary vortex tubes

advection of planetary vortex tubes

Test problem: Wind-driven circulation on a barotropic β-plane

- Square basin of width 1000 km
- Zonal wind stress (τ_0=0.1Nm^{-2})
- β = 2 x 10^{-11}m^{-1}s^{-1}



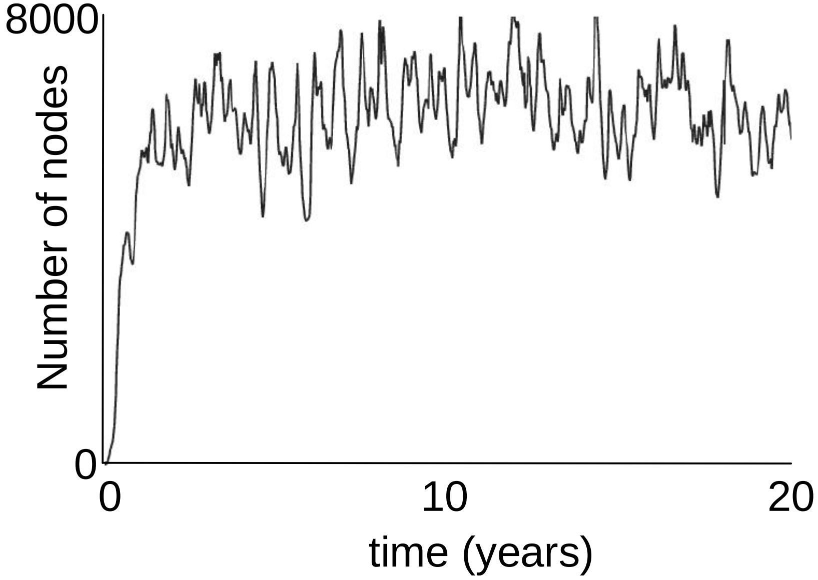
Animations from a calculation with Re = 2000 and 8000 and:

- low-order P_1-P_1 elements,
- a minimum element size of 2km and 0.5km respectively,
- a maximum element size of 50km,
- semi-implicit time step of 8 hours,
- mesh adapted once every week - based on Hessian of velocity field and specified velocity tolerances

Re=2000 Mesh and meridional velocity (red=northward; blue=southward)

QuickTime™ and a GIF decompressor are needed to see this picture.

~ 40 times less nodes than grid-points in standard finite-difference calculation with same resolution



Re=2000

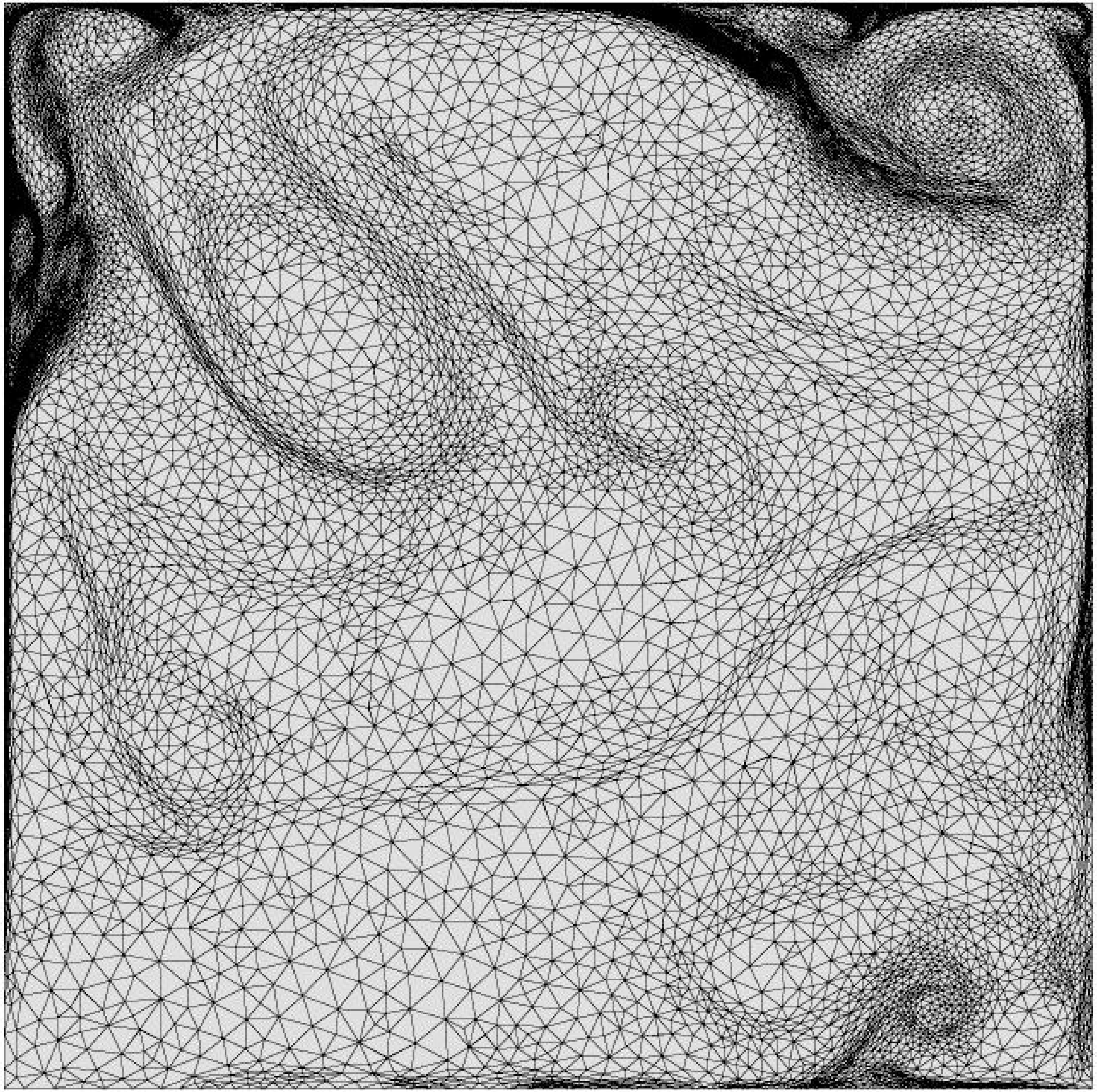
relative vorticity

QuickTime™ and a GIF decompressor are needed to see this picture.

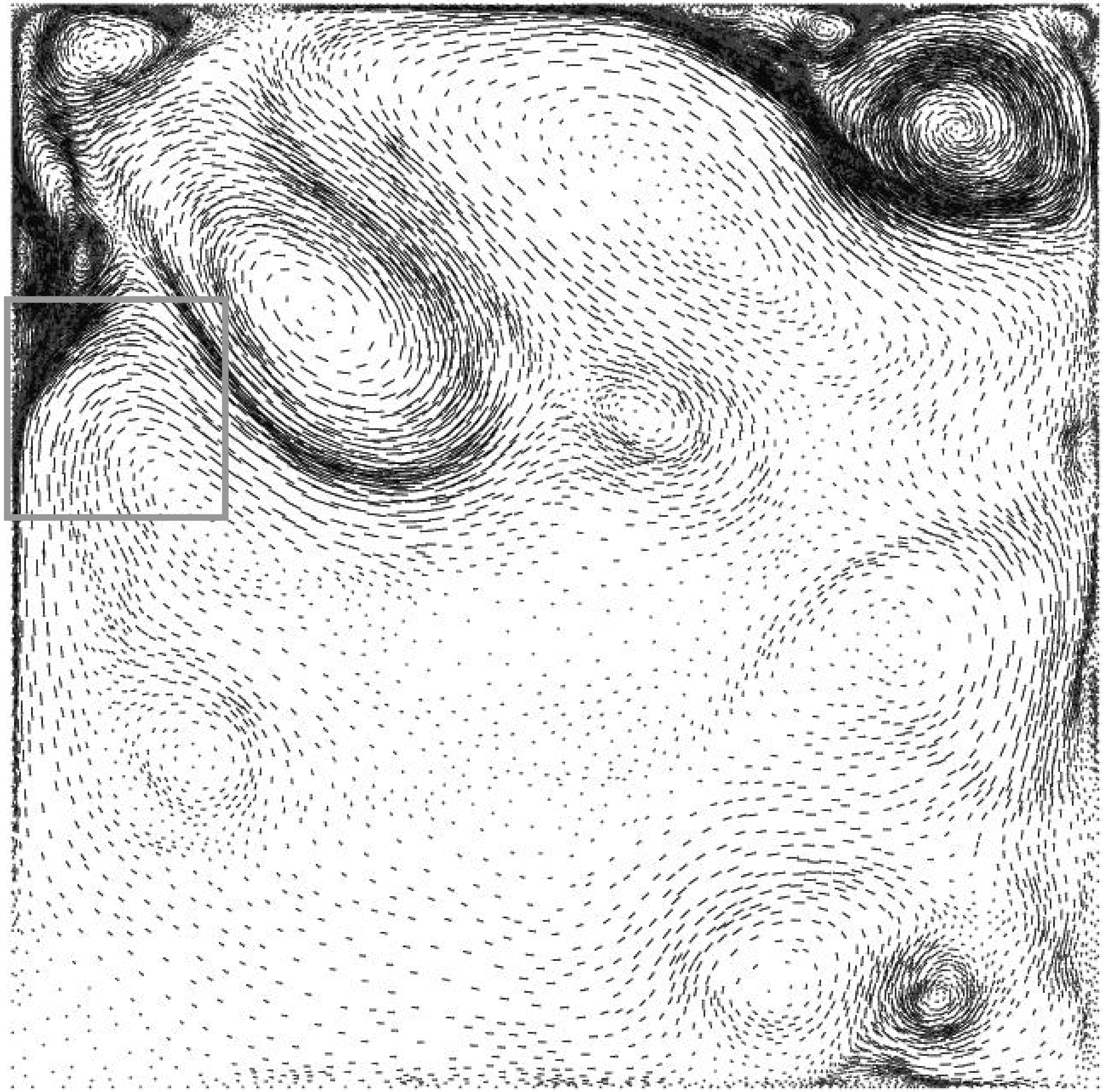
Key results from Re=2000 calculation:

- New method works well with a low-order element pair
- Can employ a relatively large time-step
- Computational overhead of additional elliptic problem is initially ~ 10%
- *However* the increased stability of the method means the solution is smoother and thus requires less elements - overall the code runs significantly faster
- Can test (statistical) convergence: for example, decreasing the minimum element size by a factor 4 leads to only a 10% increase in the number of elements

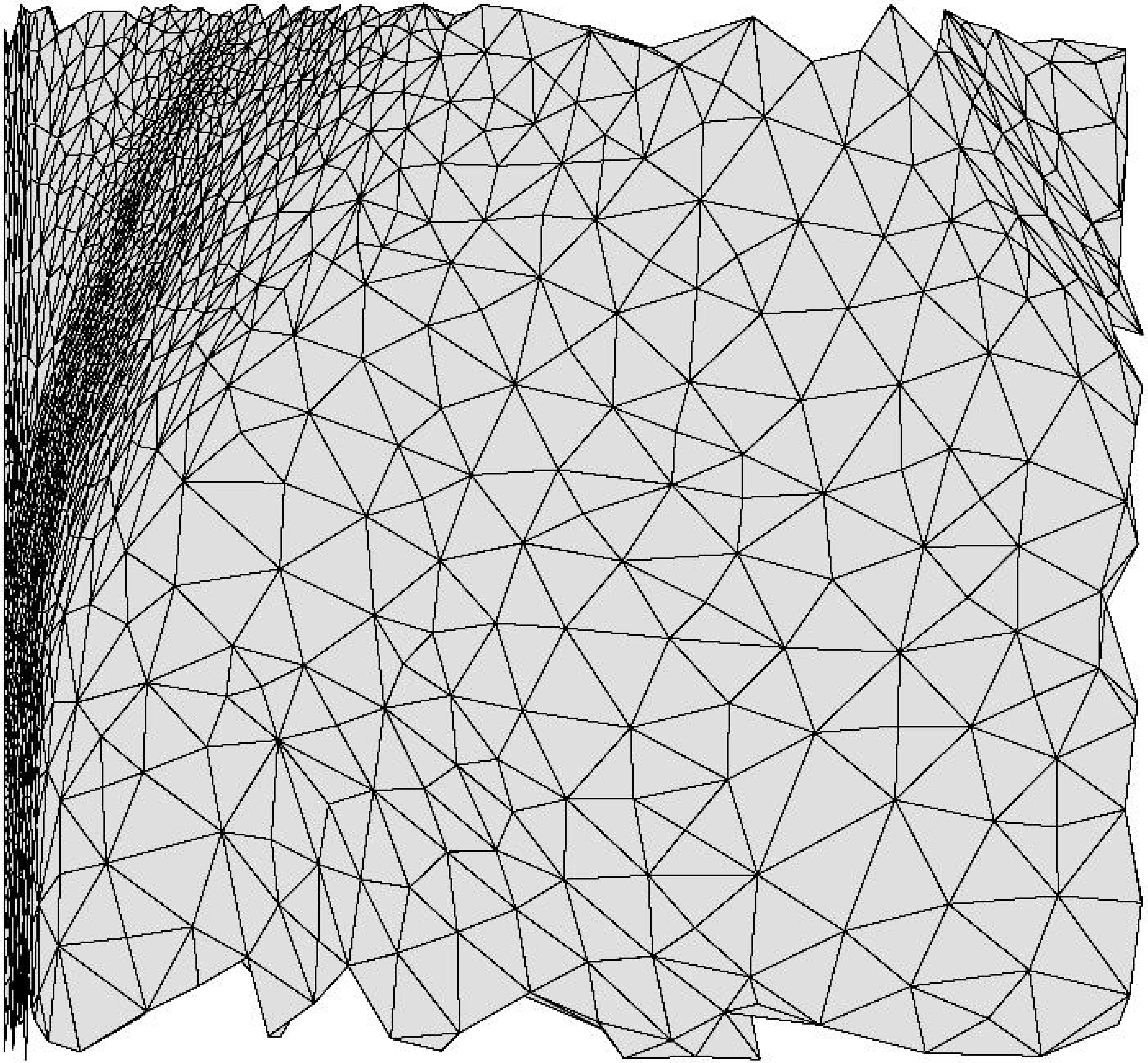
Re=8000



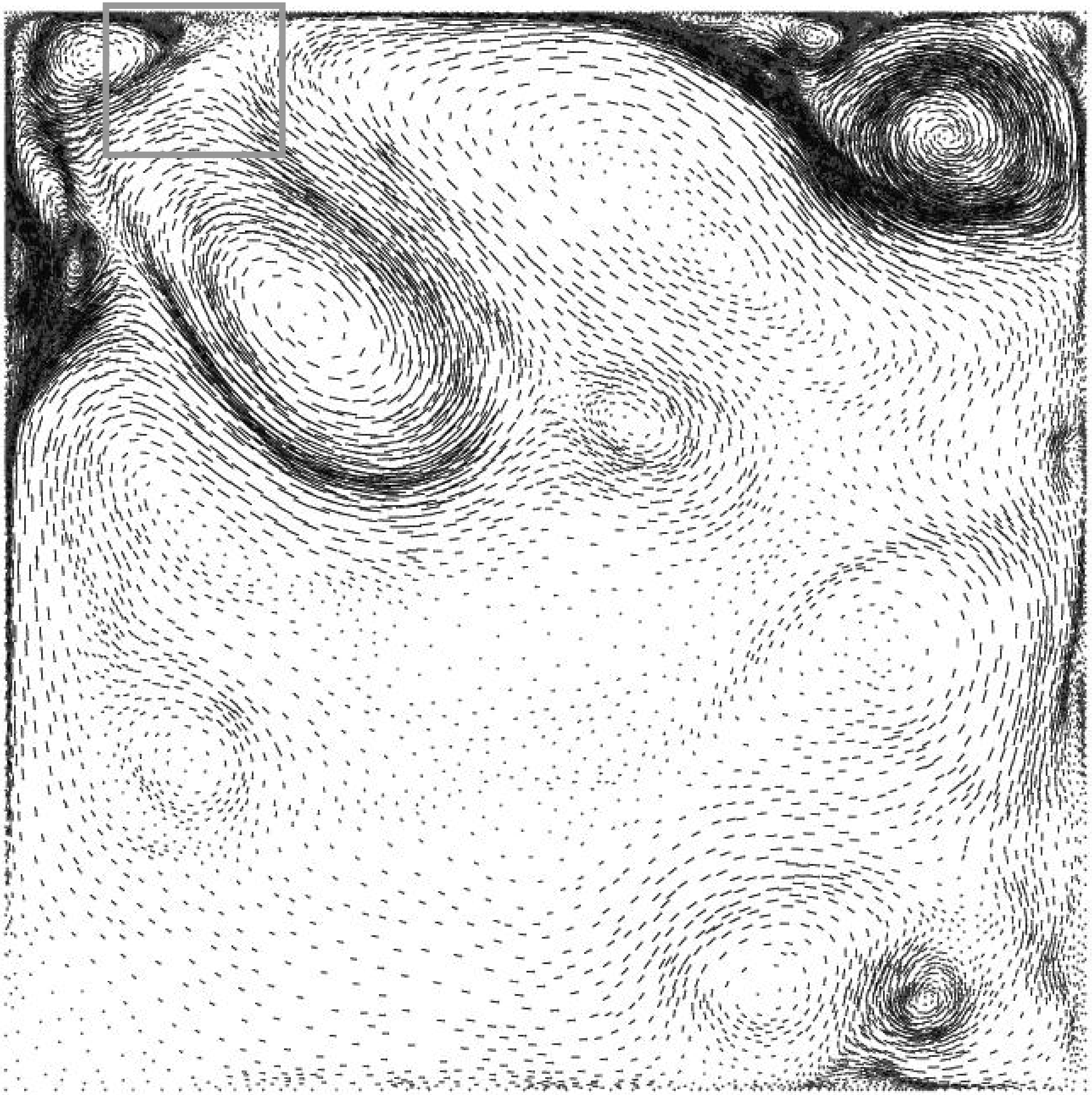
Re=8000

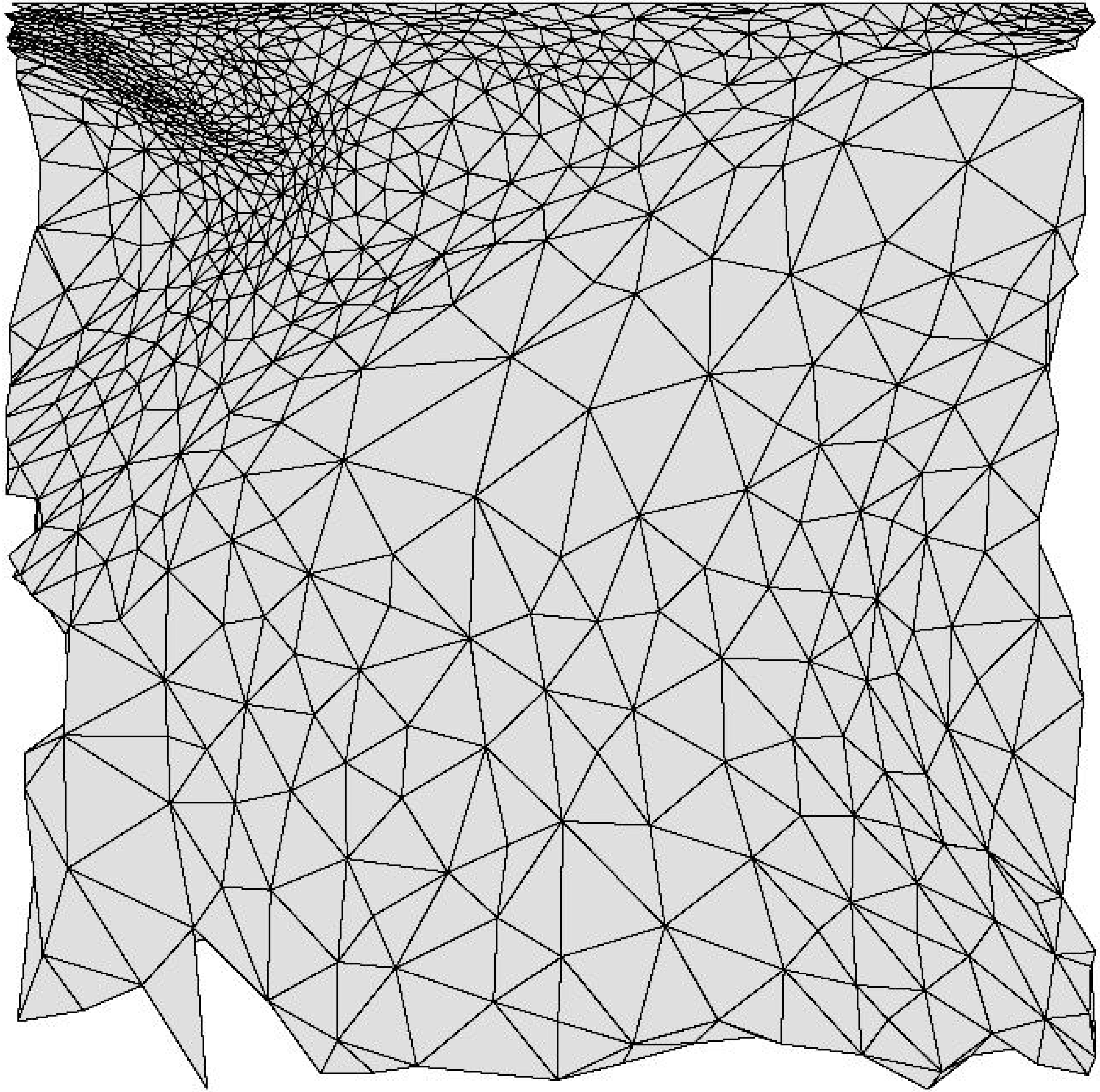


Re=8000



Re=8000





Re=8000

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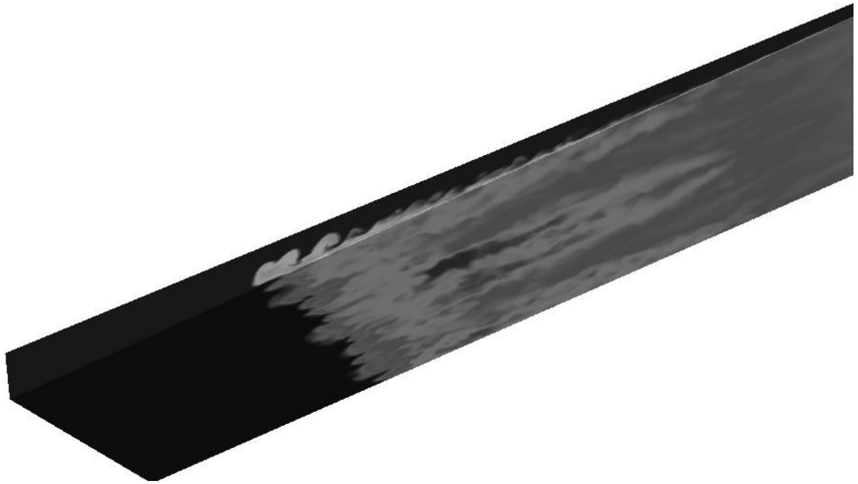
3-d Gravity Currents



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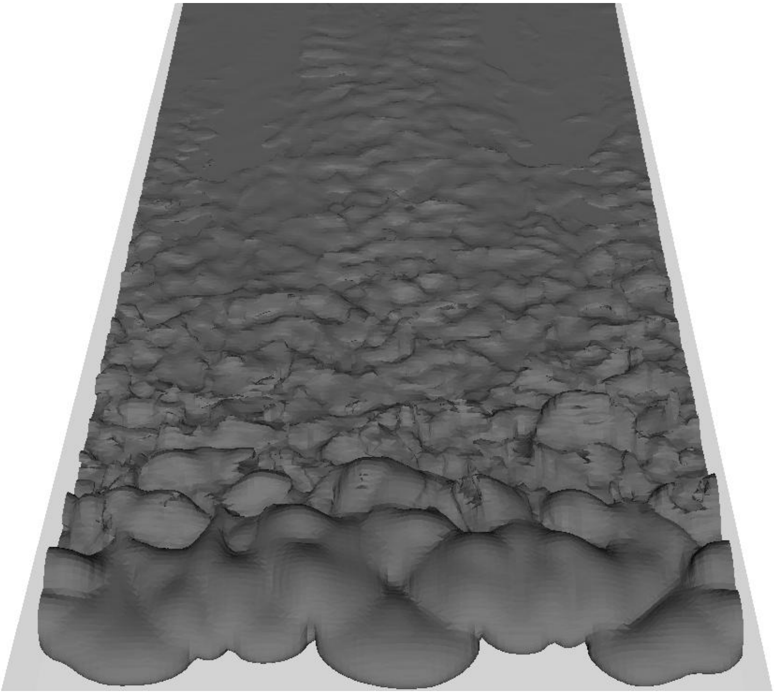
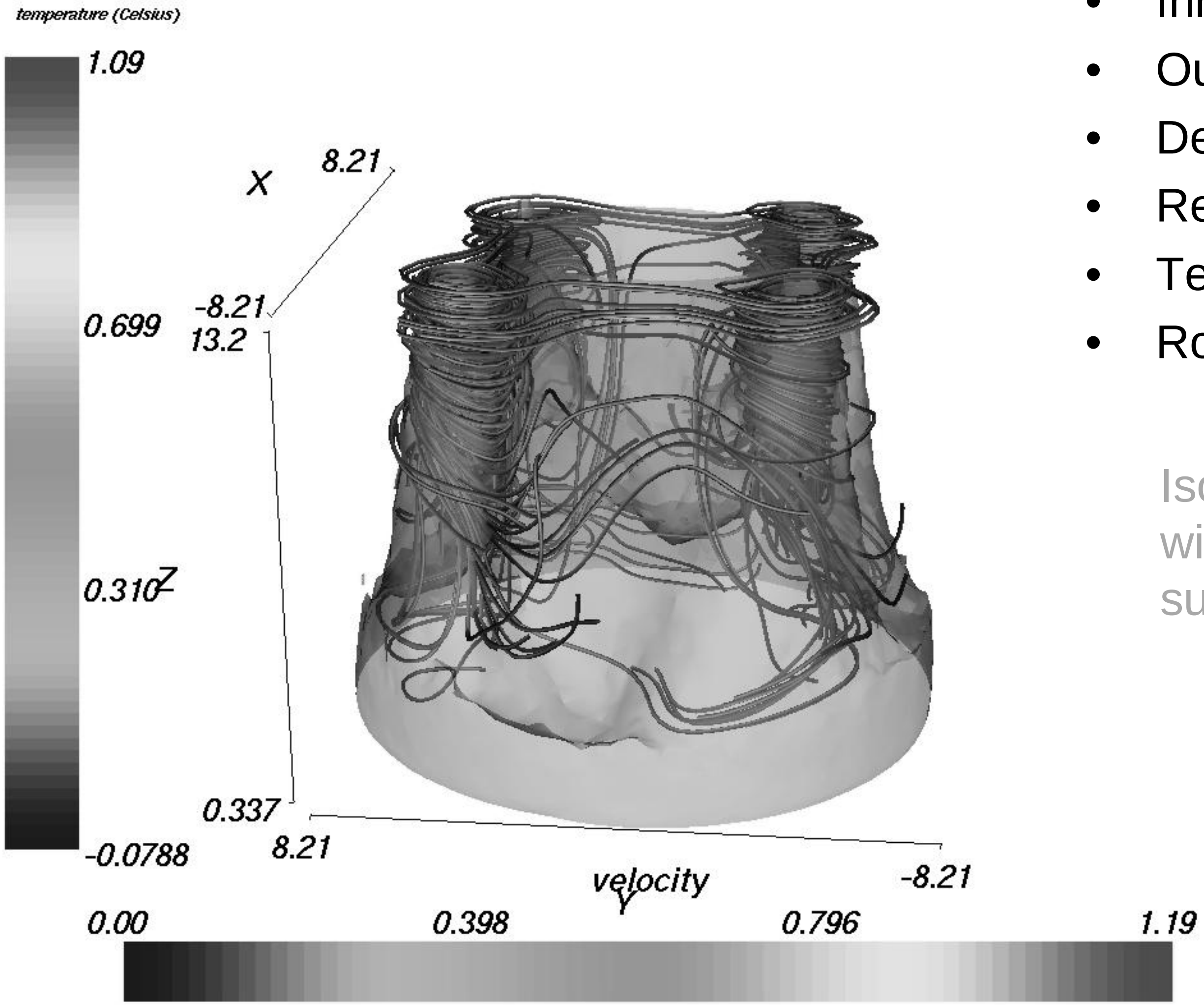


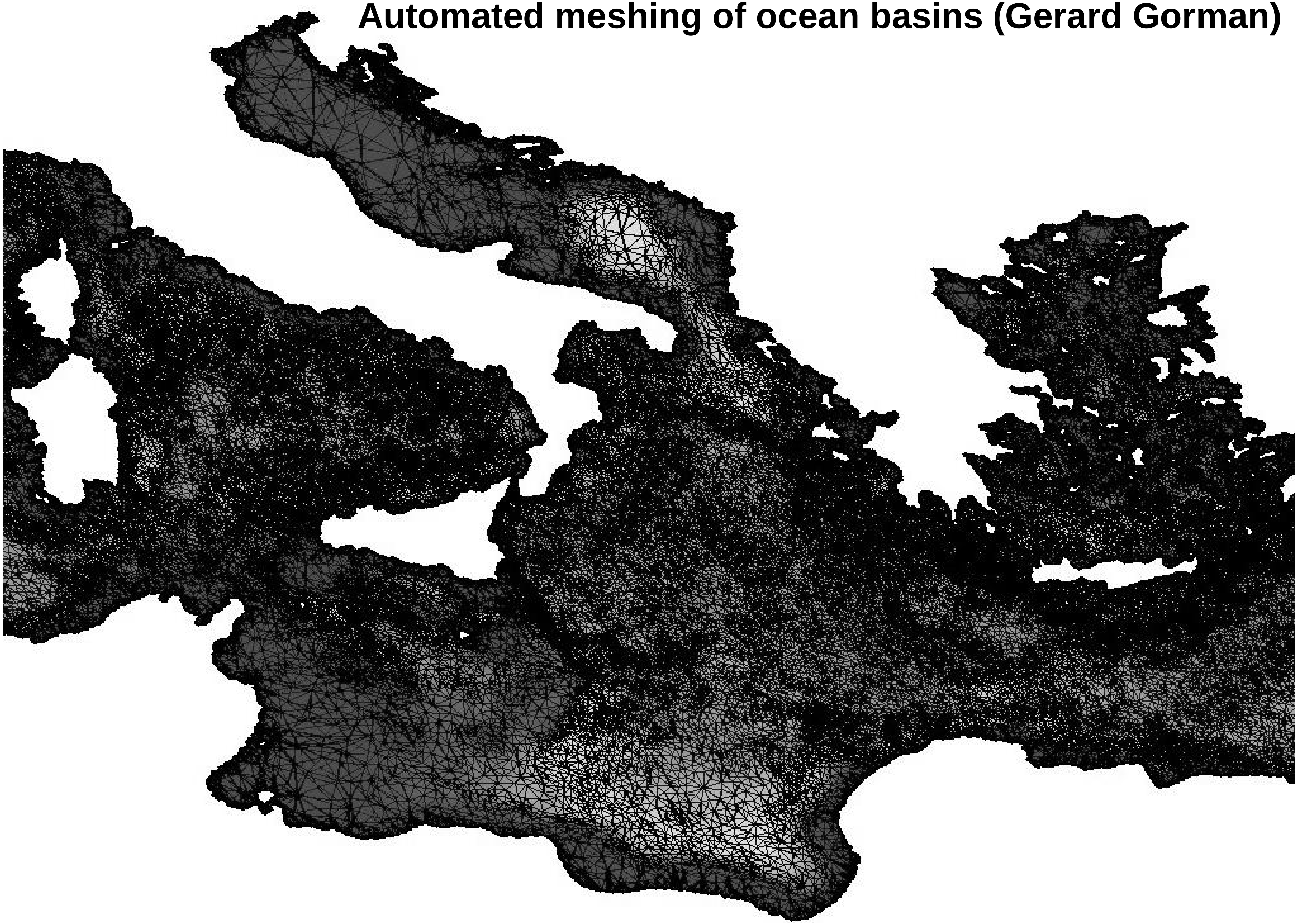
Differentially heated rotating annulus (Philip Power)

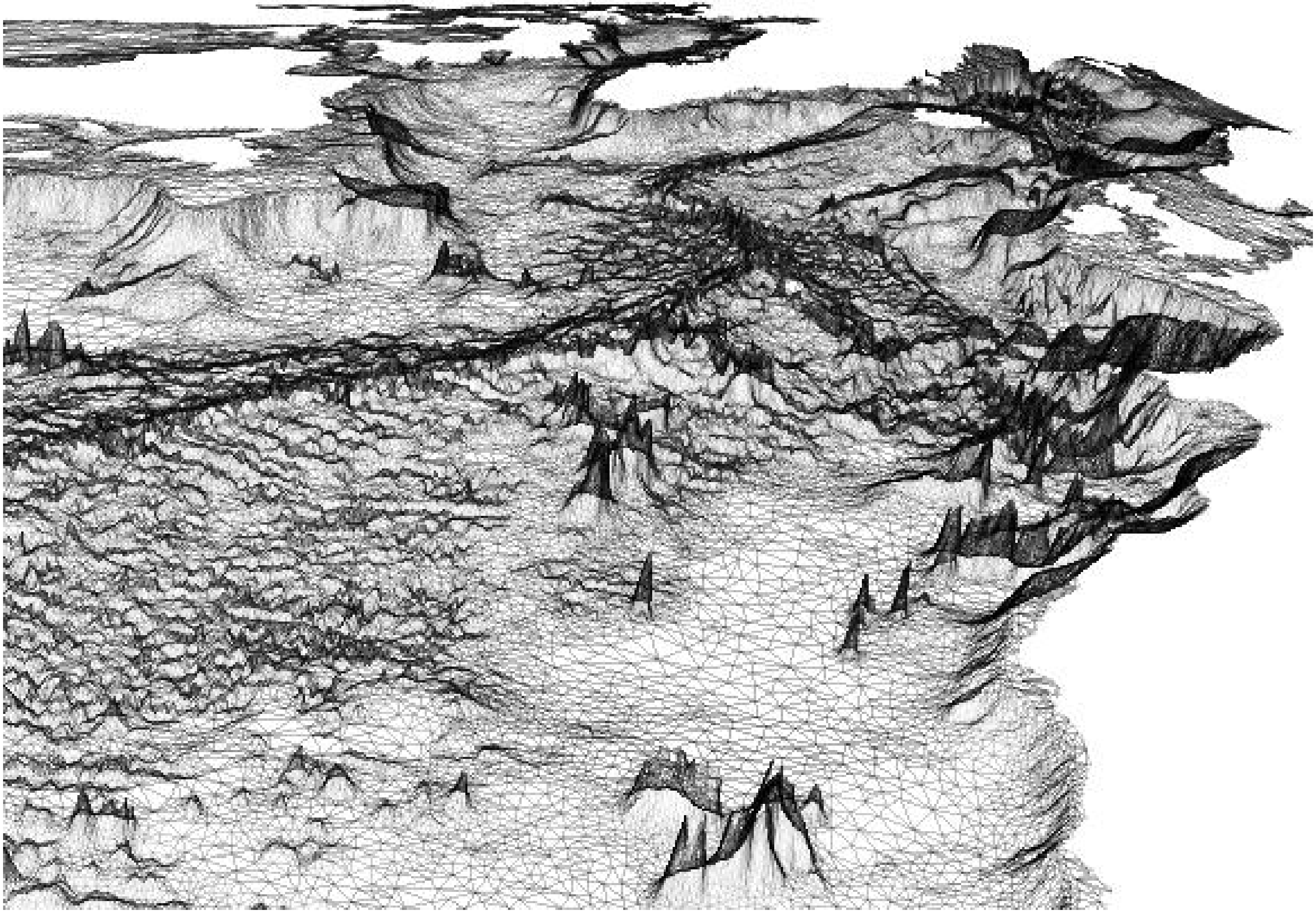


- Inner radius 4cm
- Outer radius 8.64cm
- Depth 13.5cm
- Re = 1000
- Temperature difference = 10 C
- Rotation rate = 1 – 5 rad/sec

Isosurface of temperature with streamtubes superimposed.

Automated meshing of ocean basins (Gerard Gorman)





Concluding remarks:

- We are developing a finite-element, nonhydrostatic ocean model (the “Imperial College Ocean Model” - ICOM) with unstructured, adaptive meshing.

- Early results suggest significant efficiency gains may be possible over conventional models (especially at higher resolutions).

- We remove geostrophic and hydrostatic pressures prior to integration - this allows us to employ standard element pairs.

- Some further issues being addressed:
    - low Rossby number, stratified flows;
    - refinement of error measures guiding mesh refinement (e.g., incorporate adiabatic properties by constraining the mesh to follow isopycnals in the ocean interior?)
    - free surface;
    - development of an adjoint model;
    - ……