Validity of anelastic and other equation sets as inferred from normal-mode analysis*

TERRY DAVIES†, ANDREW STANIFORTH, NIGEL WOOD,
Met Office, Bracknell, UK
AND JOHN THUBURN
University of Reading, UK

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SUMMARY

Various simplifications of the fully compressible inviscid (Euler) equations have been made over the years to filter out rapidly-propagating acoustic oscillations, resulting in various anelastic, hydrostatic and pseudo-incompressible equation sets. The principal tool used to develop approximate equation sets, and to assess their validity as a function of flow regime, has been scale analysis which has proven quite subtle to apply. Here it is shown that normal-mode analysis provides a useful complementary tool for assessing the validity of the above-mentioned approximate equation sets for both small- and large-scale flows, and leads to the following conclusions. Whilst of key importance for small-scale theoretical studies and process modelling, the anelastic equations are not recommended for either operational numerical weather prediction or climate simulation at any scale. The pseudo-incompressible set appears to be viable for numerical weather prediction, but only at short horizontal scales. For global nonhydrostatic modelling, only the fully compressible equations are suitable. Advances in numerical techniques in the past decade allow these to be integrated in a computationally efficient manner.

KEYWORDS: Pseudo-incompressible Boussinesq Nonhydrostatic Fully-compressible Global modelling

1 INTRODUCTION

A number of groups involved in running numerical weather prediction (NWP) models and/or climate models are either using or planning to use unified models. These unified models are

†Corresponding author address: Dr. Terry Davies, Met Office, London Road, Bracknell RG12 2SZ, UK

used for a wide range of scales and applications which in some cases range from high resolution mesoscale models to low resolution global climate models (typically a range of horizontal resolution of one or two orders of magnitude). The main reason for the move to unified modelling systems is the considerable cost of development and maintenance of increasingly complex and multi-functional systems. For limited-area or nesting applications there is also an important benefit in using the same model in order to reduce non-transparency effects in the application of lateral boundary conditions. To meet this multi-scale functionality, a model must be applicable over the required range of scales. Fundamental to this is the choice of equation set. Herein normal-mode analysis is used as a tool to inform this choice.

Various simplifications of the fully compressible inviscid (Euler) shallow-atmosphere equation set have been made over the years, principally for two reasons: firstly to provide theoretical insight, and secondly to reduce the computational cost of a numerical model. Regarding the latter, the limitation on timestep of an explicit time discretisation of terms responsible for the vertical propagation of acoustic modes is particularly restrictive because of the large propagation speed coupled with a typically higher vertical than horizontal resolution. This has led to various methods for filtering vertically-propagating acoustic modes from the full equations. A particularly good way of doing this, provided nonhydrostatic effects are unimportant (generally believed to be the case for horizontal scales greater than approximately 10 km, see e.g. Daley (1988)), is to only neglect the vertical acceleration term in the momentum equation. This is the basis of the hydrostatic primitive equations. However for mesoscale flows, where nonhydrostatic effects may be important, an alternative approach is required.

One such approach is to filter all the acoustic modes by replacing the mass continuity equation by the incompressibility assumption $\nabla \cdot \mathbf{u} = 0$, where $\mathbf{u}$ is the three-dimensional velocity vector. This, together with the assumption that the basic-state density $\overline{\rho}$ is independent of height, is the basis of the Boussinesq approximation (Tritton (1988) and Mahrt (1986) discuss the derivation and assumptions underpinning this approximation). The assumption that $\overline{\rho}$ is independent of height unfortunately limits validity of the Boussinesq equation set to relatively shallow flows, e.g. within the planetary boundary layer. Thunis & Bornstein (1996) present a review of hierarchies of atmospheric motions together with some approximate equation sets appropriate to each.

To reduce the severity of the Boussinesq limitation, Ogura & Phillips (1962), using a rigorous scale analysis, introduced a partially compressible equation set termed "anelastic". This equation set was used, for example, by Clark & Peltier (1977). In this approach, the basic-state is isentropic and $\overline{\rho}$ is allowed to vary in height subject to the constraint that $\nabla \cdot (\overline{\rho}\mathbf{u}) = 0$. However, it is still not valid for phenomena (e.g. deep convection or stratospheric gravity waves) which take place in an environment that is not well approximated by an isentropic basic state. In an attempt to circumvent this difficulty, Wilhelmson & Ogura (1972) allowed the basic-state potential temperature to vary in the vertical. Unfortunately though, energy conservation is then lost. As noted by Durran (1989), and using his terminology, this *modified* anelastic system has been widely used.

Adopting the rigorous scale analysis approach of Ogura & Phillips (1962), Lipps & Hemler (1982) obtained a further modified anelastic system that also permits a vertically varying (albeit rather slowly: Nance & Durran (1994)) basic-state potential temperature $\overline{\theta}$, but which achieves the Wilhelmson & Ogura (1972) goal without loss of energy conservation. Bannon (1996) also derived the same prognostic equation set via a somewhat different approach that slaves the thermodynamics to the dynamics rather than vice versa. The resulting equation set differs from that of Lipps & Hemler (1982) only in the diagnosis of the density perturbations. Bannon (1996) termed his equations the *hybrid* anelastic set as they are a hybrid of the Lipps & Hemler (1982) and Dutton & Fichtl (1969) equations. These variants of the anelastic equation set have been subsequently adopted by a number of authors for numerical models, for example Shutts & Gray (1994) and Smolarkiewicz, Margolin & Wyszogrodzki (2001).

A final variation is that of Durran (1989) who argued that the usefulness of the Lipps & Hemler (1982) anelastic system may nevertheless still be limited in regions of strong static stability (e.g. in the stratosphere) by the slow vertical variation of the basic-state potential temperature. This motivated development of his *pseudo-incompressible* equation set for which $\nabla \cdot \left(\overline{\rho}\overline{\theta}\mathbf{u}\right) = 0$ and energy is conserved.

The basis of these anelastic equation sets lies in scale analysis. However, as noted by Bryan & Fritsch (2002), the neglect of small terms can sometimes be surprisingly important. The goals of the present paper are to: perform a normal-mode analysis of the fully compressible equations and various partially compressible and incompressible approximations to them; examine possible distortion of the vertical normal modes of these approximate equation sets (with a focus on anelastic ones) and thereby infer some limitations on the validity of approximate equation sets; and examine the viability of global anelastic modelling.

The plan of the paper is as follows. The fully compressible two-dimensional $(x - z)$ Euler equations are given in Section 2. In Section 3 these are then expanded, without approximation, about a hydrostatically balanced basic state, with switches appended to various terms: the fully-compressible equations are recovered when all switches are set to unity, with various approximate equation sets resulting if some are instead set to zero. These switched equations are then linearised in Section 4 about a motionless hydrostatically balanced basic state. Normal-mode solutions of the switched linearised equations are given in Section 5 for an isothermal basic state. In Section 6 spurious distortion of the vertical structure of the normal modes, which occurs for each approximate equation set considered, is discussed. Finally, conclusions of the study are presented in Section 7.

2 FULLY COMPRESSIBLE EQUATIONS

To facilitate the analysis, it is convenient to work on a Cartesian f-plane. [Since, for shallow atmospheres, the normal mode vertical structure is independent of the horizontal geometry (Thuburn, Wood & Staniforth 2002*a*), the results will also be relevant to the spherical geometry case. It is possible to include the Coriolis terms associated with the horizontal component of the Earth's rotation. However, for the fully compressible equations, the results are significantly different from those on the f-plane only in three extreme regimes: long-zonal wavelength internal acoustic modes, extremely deep gravity modes, and extremely shallow gravity modes (Thuburn, Wood & Staniforth 2002*b*).]

The fully compressible, inviscid, vertical slice $(x - z)$ Euler equation set, in the absence of diabatic forcing, examined herein is, in standard notation,

$$\frac{Du}{Dt} + c_p\theta\frac{\partial\pi}{\partial x} - fv = 0, \tag{2.1}$$

$$\frac{Dv}{Dt} + fu = 0, \tag{2.2}$$

$$\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi}{\partial z} + g = 0, \tag{2.3}$$

$$\left(\frac{1-\kappa}{\kappa}\right)\frac{D\pi}{Dt} + \pi\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = 0, \tag{2.4}$$

$$\frac{D\theta}{Dt} = 0, \tag{2.5}$$

where

$$\pi^{(1-\kappa)/\kappa} = \frac{R}{p_{00}}\rho\theta, \tag{2.6}$$

$$\pi \equiv \left(\frac{p}{p_{00}}\right)^{\kappa}, \ \kappa \equiv R/c_p, \ R \equiv c_p - c_v, \tag{2.7}$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z}. \tag{2.8}$$

Eqs. (2.1) - (2.8) are, respectively, three components of the momentum equation, a derived pressure equation that replaces the continuity equation, the thermodynamic equation, the ideal gas law, the definition of Exner pressure, and the definition of the substantive derivative.

3 SWITCHABLE FORMS OF THE FULL EQUATIONS

Expanding, without approximation, the thermodynamic variables θ and π about a hydrostatically balanced reference state given by $\overline{\theta} = \overline{\theta}(z)$ and $\overline{\pi} = \overline{\pi}(z)$ with

$$c_p\overline{\theta}\frac{\partial\overline{\pi}}{\partial z} = -g, \tag{3.1}$$

gives

$$\frac{Du}{Dt} + c_p\left(\overline{\theta} + \delta_E\theta'\right)\frac{\partial\pi'}{\partial x} - fv = 0, \tag{3.2}$$

$$\frac{Dv}{Dt} + fu = 0, \tag{3.3}$$

$$\delta_V\frac{Dw}{Dt} + c_p\left(\overline{\theta} + \delta_E\theta'\right)\frac{\partial\pi'}{\partial z} + (1-\delta_B)\,c_p\frac{d\overline{\theta}}{dz}\pi' - \frac{g}{\overline{\theta}}\theta' = 0, \tag{3.4}$$

$$\frac{\delta_A}{\overline{\pi}}\left[\left(\frac{1-\kappa}{\kappa}\right)\frac{D}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right]\pi' + \frac{\partial u}{\partial x} + \left(\frac{\partial}{\partial z} + \frac{\delta_C}{\overline{\rho}}\frac{d\overline{\rho}}{dz} + \frac{\delta_D}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\right)w = 0, \tag{3.5}$$

$$\frac{D\left(\overline{\theta} + \theta'\right)}{Dt} = 0, \tag{3.6}$$

where θ' and π' are the perturbations (not necessarily small) about the hydrostatic reference state, and the basic-state hydrostatic relation (3.1) has been used to obtain (3.4).

Six switches (δ_V, δ_A, δ_B, δ_C, δ_D and δ_E) have been introduced into (3.4) - (3.5). When they are all set to unity, the fully compressible equation set results. When some of them

Equation set	δ_V	δ_A	δ_B	δ_C	δ_D	δ_E
Fully compressible	1	1	1	1	1	1
Hydrostatic	0	1	1	1	1	1
Pseudo-incompressible (Durran 1989)	1	0	1	1	1	1
Anelastic (Wilhelmson & Ogura 1972)	1	0	1	1	0	0
Anelastic (Lipps & Hemler 1982)	1	0	0	1	0	0
Boussinesq	1	0	1	0	0	0

Table 1: Switch settings for various equation sets.

are set to unity while others are set to zero, various simplified partially-compressible and incompressible equation subsets (hydrostatic, pseudo-incompressible, anelastic, Boussinesq) then result - see Table 1 for details. The validity of these approximations then requires the thermodynamic perturbations to be small in some sense (see the cited references for details).

There are various points to note. For all approximate nonhydrostatic sets (i.e. anelastic, Boussinesq and pseudo-incompressible), $\delta_A \equiv 0$. Of these, the pseudo-incompressible set of Durran (1989) is unique in that it does not linearise the pressure gradient terms. The anelastic set of Wilhelmson & Ogura (1972) is an extension of the equations of Ogura & Phillips (1962). This latter set can be obtained from that of Wilhelmson & Ogura (1972) by restricting the reference profile to be isentropic. At the level of detail needed to derive the results presented herein, the anelastic set of Lipps & Hemler (1982) is equivalent to the equation sets of Dutton & Fichtl (1969) (with (3.6) replaced by $D\left(\theta'/\overline{\theta}\right)/Dt+\left(w/\overline{\theta}\right)d\overline{\theta}/dz = 0$, which is equivalent to (3.6) to linear order) and Bannon (1996). The term proportional to $(1-\delta_B)$ has been added to the full equation set in order to incorporate the particular, energy conserving, approximation of this family of anelastic sets. Finally, the Boussinesq set requires all undifferentiated reference state variables to be replaced by constant reference values, independent of height (e.g. Bannon (1996)).

4 SWITCHED LINEARISED EQUATIONS

Consider a motionless hydrostatically balanced basic state on an f-plane

$$\overline{u} = \overline{v} = \overline{w} = 0, \overline{\pi} = \overline{\pi}\left(z\right), \overline{\rho} = \overline{\rho}\left(z\right), \overline{\theta} = \overline{\theta}\left(z\right), \tag{4.1}$$

that satisfies (3.1).

Expanding (3.2) - (3.6) about this basic state and linearising then gives

$$\frac{\partial u}{\partial t} + c_p\overline{\theta}\frac{\partial \pi}{\partial x} - fv = 0, \tag{4.2}$$

$$\frac{\partial v}{\partial t} + fu = 0, \tag{4.3}$$

$$\delta_V\frac{\partial w}{\partial t} + c_p\overline{\theta}\frac{\partial \pi}{\partial z} + \left(1-\delta_B\right)c_p\frac{d\overline{\theta}}{dz}\pi - \frac{g}{\overline{\theta}}\theta = 0, \tag{4.4}$$

$$\frac{\delta_A}{\overline{\pi}}\left(\frac{1-\kappa}{\kappa}\right)\frac{\partial \pi}{\partial t}+\frac{\partial u}{\partial x}+\left(\frac{\partial}{\partial z}+\frac{\delta_C}{\overline{\rho}}\frac{d\overline{\rho}}{dz}+\frac{\delta_D}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\right)w=0, \tag{4.5}$$

$$\frac{\partial \theta}{\partial t}+\frac{d\overline{\theta}}{dz}w=0, \tag{4.6}$$

where the dependent variables are understood to be perturbation quantities (now assumed small). Note that the δ_E switch is now absent - it multiplies a nonlinear (product of perturbations) term which is then neglected as a consequence of the linearisation.

5 NORMAL-MODE ANALYSIS

Now further choose the basic-state to be isothermal in order to be able to find analytic solutions, so that

$$\left.\begin{array}{c}\overline{u}=\overline{v}=\overline{w}=0,\\ \overline{T}=constant,\\ \overline{\theta}\left(z\right)=\overline{\theta_S}e^{\frac{\kappa z}{H}},\ \overline{\rho}\left(z\right)=\overline{\rho_S}e^{-\frac{z}{H}},\ \overline{\pi}\left(z\right)=\overline{\pi_S}e^{-\frac{\kappa z}{H}},\end{array}\right\} \tag{5.1}$$

where

$$\overline{\rho_S}=\frac{p_{00}}{R\overline{T}}\overline{\pi_S}^{1/\kappa},\ \overline{\theta_S}=\frac{\overline{T}}{\overline{\pi_S}}, \tag{5.2}$$

subscript "S" denotes evaluation at the surface, and

$$H\equiv\frac{R\overline{T}}{g}, \tag{5.3}$$

is the scale height of the atmosphere. [The numerical normal mode calculations of Thuburn et al. (2002a) for a spherical rotating atmosphere have been extended to compare results for an isothermal basic state with those for a U.S. Standard Atmosphere basic state. It was found that the structures and frequencies of the well-resolved normal modes differ only slightly for these two atmospheres, implying that the general conclusions drawn below for an isothermal basic state should also carry over to more realistic ones.]

To proceed, a choice of upper boundary condition has to be made. However, there is no universally-accepted way of closing the problem at the top of the atmosphere. Here, for analytical tractability, rigid lower and upper boundaries at $z=0$ and $z=z_T=$ constant are assumed. Because of the controlled way in which the different equation sets are compared, it is unlikely that a different choice of upper boundary condition, which is applied in exactly the same way to all examined equation sets, would materially change the conclusions. In particular, it is unlikely that a different choice of upper boundary condition would remedy the inaccuracies seen in the approximate equation sets. The normal modes of (4.2) - (4.6) for the above isothermal basic-state atmosphere and the assumed boundary conditions have therefore been determined following Section 3 of Thuburn et al. (2002b). They fall into two classes: external and internal modes.

5.1 External mode

The external mode is

$$u = u_0 \exp\left[i\left(k_x x - \sigma t\right)\right] \exp\left(\frac{\delta_B \kappa z}{H}\right), \tag{5.4}$$

$$v = v_0 \exp\left[i\left(k_x x - \sigma t\right)\right] \exp\left(\frac{\delta_B \kappa z}{H}\right), \tag{5.5}$$

$$\pi = \pi_0 \exp\left[i\left(k_x x - \sigma t\right)\right] \exp\left[\left(\delta_B - 1\right)\frac{\kappa z}{H}\right], \tag{5.6}$$

$$w = 0, \tag{5.7}$$

$$\theta = 0, \tag{5.8}$$

where

$$u_0 = \frac{\sigma k_x c_p \overline{\theta_S}}{\sigma^2 - f^2}\pi_0, \tag{5.9}$$

$$v_0 = \frac{-i f k_x c_p \overline{\theta_S}}{\sigma^2 - f^2}\pi_0, \tag{5.10}$$

$$\sigma\left[\delta_A\left(\sigma^2 - f^2\right) - c_s^2 k_x^2\right] = 0, \tag{5.11}$$

π_0 is an arbitrary constant that determines the amplitude of the Exner pressure perturbation, and $c_s^2 \equiv R\overline{T}/\left(1-\kappa\right) \equiv gH/\left(1-\kappa\right)$ is the reference state sound speed.

For the Boussinesq equation set, since the undifferentiated basic-state variables are constant, the above isothermal analysis is not applicable. However it can be shown that the external mode is stationary (i.e. $\sigma = 0$), with $u = 0$, arbitrary vertical variation for π, and v and θ satisfy

$$c_p\overline{\theta}\frac{\partial \pi}{\partial x} - f v = 0, \tag{5.12}$$

$$c_p\overline{\theta}\frac{\partial \pi}{\partial z} - \frac{g}{\overline{\theta}}\theta = 0, \tag{5.13}$$

where $\overline{\theta}$ is constant.

Eq. (5.11) is the dispersion relation for an external mode with, in general, three roots. The first, $\sigma = 0$, is the degenerate stationary external Rossby mode on an f-plane. Provided $\delta_A \neq 0$, there are also two external acoustic or Lamb modes (cf. (4.4) of Thuburn et al. (2002*b*)) for which

$$\sigma^2 = f^2 + c_s^2 k_x^2. \tag{5.14}$$

However if $\delta_A = 0$, then these modes are identically filtered from the equation set.

5.2 Internal modes

The internal modes are

$$u = u_0 \exp\left[i\left(k_x x - \sigma t\right)\right]\left[\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)\right] \exp\left\{\left[\frac{\delta_C + \left(\delta_B - \delta_D\right)\kappa}{2H}\right] z\right\}, \tag{5.15}$$

$$v = v_0 \exp\left[i\left(k_x x - \sigma t\right)\right]\left[\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)\right] \exp\left\{\left[\frac{\delta_C + \left(\delta_B - \delta_D\right)\kappa}{2H}\right] z\right\}, \tag{5.16}$$

$$\pi = \pi_0 \exp\left[i\left(k_x x - \sigma t\right)\right]\left[\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)\right] \exp\left\{\left[\frac{\delta_C + \left(\delta_B - \delta_D - 2\right)\kappa}{2H}\right] z\right\}, \quad (5.17)$$

$$w = w_0 \exp\left[i\left(k_x x - \sigma t\right)\right] \sin\left(k_z z\right) \exp\left\{\left[\frac{\delta_C + \left(\delta_B - \delta_D\right)\kappa}{2H}\right] z\right\}, \quad (5.18)$$

$$\theta = \theta_0 \exp\left[i\left(k_x x - \sigma t\right)\right] \sin\left(k_z z\right) \exp\left\{\left[\frac{\delta_C + \left(\delta_B - \delta_D + 2\right)\kappa}{2H}\right] z\right\}, \quad (5.19)$$

where

$$\Gamma\left(\delta_B, \delta_C, \delta_D\right) \equiv \frac{\delta_C - \left(\delta_B + \delta_D\right)\kappa}{2H}, \quad (5.20)$$

$$u_0 = \frac{\sigma k_x c_p \overline{\theta_S}}{\sigma^2 - f^2} \pi_0, \quad (5.21)$$

$$v_0 = \frac{-i f k_x c_p \overline{\theta_S}}{\sigma^2 - f^2} \pi_0, \quad (5.22)$$

$$w_0 = \frac{i \sigma c_p \overline{\theta_S}\left(k_z^2 + \Gamma^2\right)}{N^2 - \delta_V \sigma^2} \pi_0, \quad (5.23)$$

$$\theta_0 = \frac{g \overline{\theta_S}\left(k_z^2 + \Gamma^2\right)}{\overline{\pi_S}\left(N^2 - \delta_V \sigma^2\right)} \pi_0, \quad (5.24)$$

and

$$\sigma\left\{\left[\delta_A\left(\sigma^2 - f^2\right) - c_s^2 k_x^2\right]\left(\delta_V \sigma^2 - N^2\right) - c_s^2\left(k_z^2 + \Gamma^2\right)\left(\sigma^2 - f^2\right)\right\} = 0. \quad (5.25)$$

In the above, $N^2 \equiv g d \ln \overline{\theta}/dz \equiv g^2/\left(c_p \overline{T}\right) \equiv \kappa g/H$ is the reference state buoyancy frequency, and $k_z \equiv m\pi/z_T$ where m is a positive integer and $z = z_T$ defines the assumed rigid lid. [Note that wherever π is associated with k_z, as in its definition, it is exceptionally not the Exner function defined by (2.8), but the ratio of the circumference of a circle to its diameter.]

For the Boussinesq equation set, since the undifferentiated basic-state variables are constant, the above isothermal analysis is not strictly applicable. However it can be shown that, for $N^2 \equiv g d \ln \overline{\theta}/dz =$ constant, the above results hold provided $1/H$ is everywhere replaced by zero (which corresponds to an infinite height scale) except that is in the expression for N^2.

Writing the expressions for u, v and π for the internal modes in the form of (5.15) - (5.17) facilitates comparison with the corresponding expressions of Thuburn et al. (2002*b*). For the discussion and interpretation of the following section it is however convenient to rewrite $\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right)$ as

$$\Gamma \sin\left(k_z z\right) - k_z \cos\left(k_z z\right) \equiv -\left(k_z^2 + \Gamma^2\right)^{1/2} \cos\left(k_z z + \Delta\right), \quad (5.26)$$

where $\Delta = \mathrm{Tan}^{-1}\left(\Gamma/k_z\right)$ is a phase and Tan^{-1} denotes the principal value of $\tan^{-1}$. The factor $\left(k_z^2 + \Gamma^2\right)^{1/2}$ can then be absorbed in the definition of π_0, and w_0 and θ_0 redefined as

$$w_0 = \frac{i \sigma c_p \overline{\theta_S}\left(k_z^2 + \Gamma^2\right)^{1/2}}{N^2 - \delta_V \sigma^2} \pi_0, \quad (5.27)$$

$$\theta_0 = \frac{g\overline{\theta_s}\left(k_z^2+\Gamma^2\right)^{1/2}}{\overline{\pi_s}\left(N^2-\delta_V\sigma^2\right)}\pi_0. \tag{5.28}$$

Eq. (5.25) is the dispersion relation for internal modes and, when $\delta_V \equiv \delta_A \equiv \delta_B \equiv \delta_C \equiv \delta_D \equiv 1$, it corresponds to (4.5) of Thuburn et al. (2002*b*) after correction of a typographic error in this latter equation [the factor $(\sigma_0^2 - f^2)^2$ in the last term should be $(\sigma_0^2 - f^2)$]. In general there are five roots. The first, $\sigma = 0$, is the degenerate stationary internal Rossby mode on an f-plane. The remaining quartic in σ is a quadratic in σ^2, and its solutions correspond to pairs of internal acoustic and gravity modes. If $\delta_V\delta_A = 0$, then the internal acoustic modes are identically filtered from the equation set, and the quartic reduces to a quadratic for the frequencies of the internal gravity modes. The squared frequency of the internal gravity modes is then:

Hydrostatic $[\delta_V = 0]$

$$\sigma^2 = f^2 + \frac{k_x^2 N^2}{k_z^2 + \left[\frac{\delta_C-(\delta_B+\delta_D)\kappa}{2H}\right]^2 + \delta_A\frac{N^2}{c_s^2}}, \tag{5.29}$$

Anelastic & pseudo-incompressible $[\delta_A = 0]$

$$\sigma^2 = f^2 + \frac{k_x^2\left(N^2-\delta_V f^2\right)}{k_z^2 + \left[\frac{\delta_C-(\delta_B+\delta_D)\kappa}{2H}\right]^2 + \delta_V k_x^2}, \tag{5.30}$$

Hydrostatic, anelastic & pseudo-incompressible $[\delta_V = \delta_A = 0]$

$$\sigma^2 = f^2 + \frac{k_x^2 N^2}{k_z^2 + \left[\frac{\delta_C-(\delta_B+\delta_D)\kappa}{2H}\right]^2}. \tag{5.31}$$

6 DISCUSSION

A state vector $\underline{\Psi}$ for a normal mode may be defined as

$$\underline{\Psi} \equiv \left(u, v, \pi, w, \theta\right)^T, \tag{6.1}$$

where u, v, π, w and θ are given by (5.4) - (5.8) for an external mode, and by (5.15) - (5.19) for an internal mode.

Examination of (5.15) - (5.19) shows that spurious distortion of the vertical structure of the state vector $\underline{\Psi}$ of a given internal normal mode, induced by setting one or more switches to zero, can occur in three different ways:

1. height-scale distortion[i], i.e. distortion of the factor $\exp\left\{\left[\delta_C + \left(\delta_B-\delta_D\right)\kappa\right]z/\left(2H\right)\right\}$ common to each component of the normal-mode state vector;

2. energy redistribution, i.e. changes to the coefficient ratios $u_0 : v_0 : \pi_0 : w_0 : \theta_0$, where u_0, v_0, w_0 and θ_0 are defined in terms of π_0 by (5.21), (5.22), (5.27) and (5.28);

[i]"height-scale" here should not be confused with the scale height H

Equation set	Height-scale distortion	Energy redistribution	
		Rossby	Lamb
Fully compressible	No	No	No
Hydrostatic	No	No	No
Pseudo-incompressible (Durran 1989)	No	No	-
Anelastic (Wilhelmson & Ogura 1972)	No	No	-
Anelastic (Lipps & Hemler 1982)	Yes	No	-
Boussinesq	Yes	Yes	-

Table 2: Height-scale distortion and energy redistribution (see text for definition of these) of external modes as a function of equation set. "-" denotes Lamb modes are filtered out.

3. relocation of modal zeros, i.e. changes to the roots of

$$\cos\left(k_z z + \Delta\right) = 0, \tag{6.2}$$

or, equivalently, to the roots of

$$\tan\left(k_z z\right) = \frac{2Hk_z}{\delta_C - \left(\delta_B + \delta_D\right)\kappa}. \tag{6.3}$$

Since $\exp\left\{\left[\delta_C + \left(\delta_B - \delta_D\right)\kappa\right] z/\left(2H\right)\right\}$ is common to u, v, π, w and θ, the first way does not redistribute energy between a mode's kinetic, thermobaric and elastic components, nor does the third because it has no impact on the mode's amplitude. The second way does however - cf. (2.24) of Thuburn et al. (2002a) - and hence the adopted nomenclature.

Examination of (5.4) - (5.8) shows that only the first two of the above mechanisms are relevant to the spurious distortion of an external mode: there are no zeros for an external mode.

The existence or not of height-scale distortion, and/ or energy redistribution, and/ or relocation of zeros for external and internal modes, are respectively summarised in Tables 2 and 3 as a function of equation set. Some brief comments on spurious frequency change and mode distortion follow.

The effect on mode frequencies of various approximations to the governing equations are most noticeable for the deepest modes. As a concrete example, the internal mode frequencies have been computed using (5.25) for the different equation sets for a mid-latitude f-plane (at 45 degrees) with a reference state temperature $\overline{T} = 250$K and a rigid lid at $z_T = 80$ km. On the one hand, for a short horizontal wavelength of 10 km (see Fig. 1), all of the equation sets approximate the internal gravity mode frequencies very accurately except the hydrostatic one. This latter set overestimates the frequency by a factor 2 for a vertical wavelength of 18 km, i.e. vertical mode $m = 9$. The errors increase more rapidly for even deeper modes. This is consistent with the breakdown of the hydrostatic assumption as the aspect ratio k_x/k_z is no longer small. On the other hand, for a long horizontal wavelength of 1000 km (see Fig. 2), the hydrostatic equations and the Lipps & Hemler (1982) anelastic equations capture the frequencies of all internal gravity modes almost exactly, while the remaining schemes have

Equation set	Height-scale distortion	Energy redistribution		Relocation of zeros
		Rossby	Gravity	
Fully compressible	No	No	No	No
Hydrostatic	No	No	Yes	No
Pseudo-incompressible (Durran 1989)	No	No	Yes	No
Anelastic (Wilhelmson & Ogura 1972)	Yes	Yes	Yes	Yes
Anelastic (Lipps & Hemler 1982)	No	Yes	Yes	Yes
Boussinesq	Yes	Yes	Yes	Yes

Table 3: Height-scale distortion, energy redistribution, and relocation of zeros (see text for definition of these) of internal modes as a function of equation set.

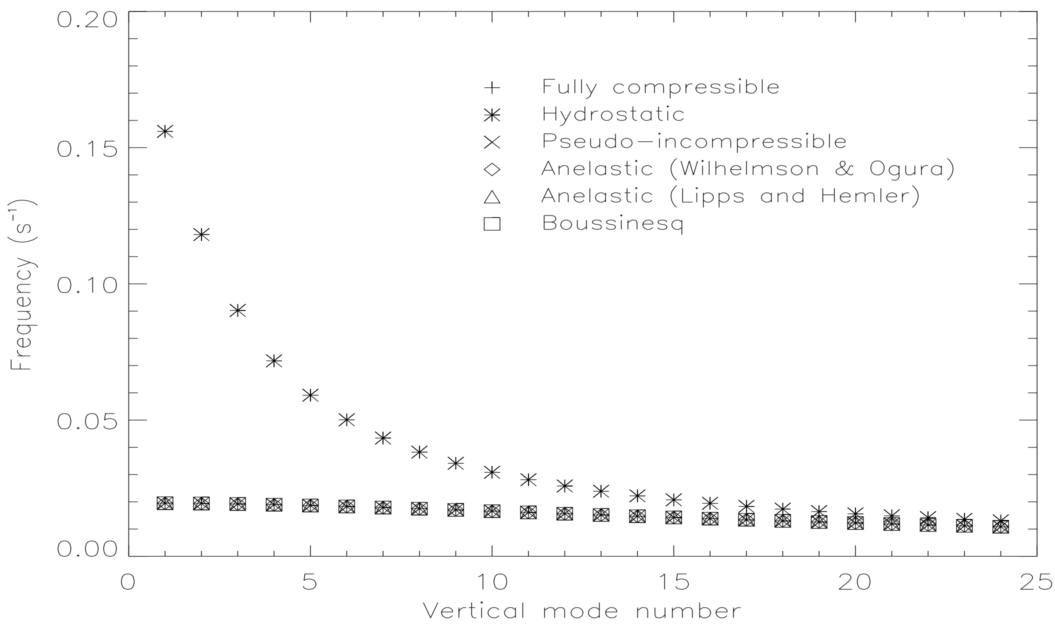

Figure 1: Frequency σ vs. vertical internal mode number m, where $k_z \equiv m\pi/z_T$ is vertical wavenumber, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and a horizontal wavelength of $2\pi/k_x = 10$ km.

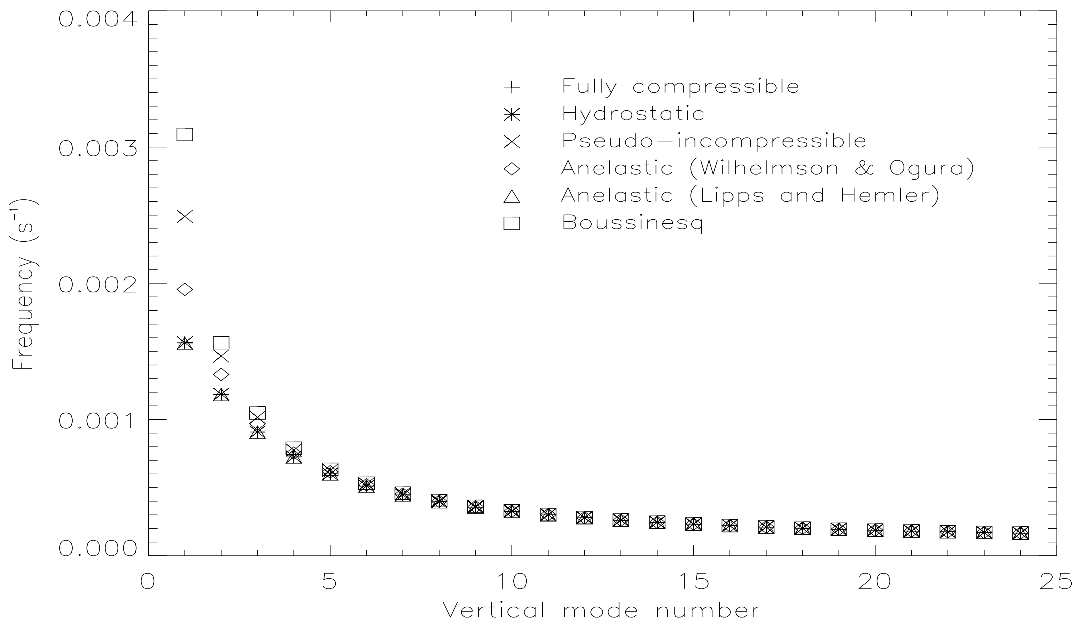

Figure 2: Same as for Fig. 1, except for a horizontal wavelength of 1000 km.

increasingly significant errors for vertical wavelengths greater than about 40 km, i.e. for $1 \leq m \leq 4$.

For the external mode, all approximate equation sets considered herein correctly capture the zero frequency of the degenerate Rossby mode. However only the hydrostatic primitive equations correctly capture the frequency of the Lamb waves: all other sets filter them out (by setting $\delta_A \equiv 0$) and would therefore be inappropriate for flows where these modes carry non-negligible energy.

The height scale of the external modes is exactly captured by all of the equation sets considered except the Lipps & Hemler (1982) set, since it is the only set that has $\delta_B = 0$, and also the Boussinesq one. For the Lipps & Hemler (1982) anelastic equations (and inter-alia those of Dutton & Fichtl (1969) and Bannon (1996)), the amplitude of the external Rossby mode is spuriously reduced by a factor of $\exp(\kappa z/H)$, e.g. by a factor of approximately 1.33 at $z = H$ with further reduction occurring for $z > H$. The effect on the height scale of internal modes depends on the coefficient $[\delta_C + (\delta_B - \delta_D)\,\kappa]$: modes are spuriously amplified by a factor $\exp\{[\delta_C + (\delta_B - \delta_D)\,\kappa - 1]\, z/(2H)\}$. For the hydrostatic, pseudo-incompressible, and Lipps & Hemler (1982) equation sets this factor is identically unity. However, for the Wilhelmson & Ogura (1972) set it equals $\exp[\kappa z/(2H)]$ (a spurious growth with height - approximately a factor of 1.15 at $z = H$), while for the Boussinesq set, noting that $1/H$ is replaced by zero, it equals $\exp[-z/(2H)]$ (a spurious lack of growth with height - approximately a factor of 0.61 at $z = H$).

Equation sets for which $\Gamma \neq (1 - 2\kappa)/(2H)$ will relocate the zeros of internal modes relative to the fully compressible case. The hydrostatic and pseudo-incompressible sets give no such relocation. Figure 3 displays Δ (which is defined below equation (5.26) and is independent of horizontal wavenumber k_x) vs. internal mode number, for the equation sets

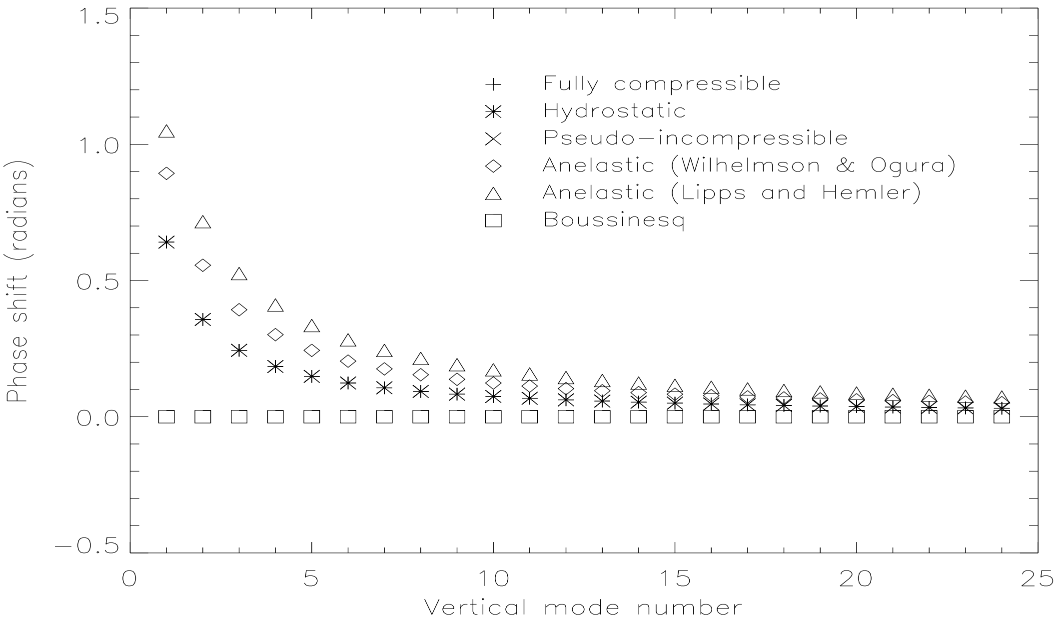

Figure 3: Phase shift Δ vs. vertical internal mode number m, where $k_z \equiv m\pi/z_T$ is vertical wavenumber, for the equation sets considered herein. Results are for a rigid lid at $z_T = 80$ km and are independent of horizontal wavelength $2\pi/k_x$. Note that the fully-compressible, hydrostatic and pseudo-incompressible points all coincide.

considered. The Boussinesq approximation gives by far the largest errors: Δ is identically and spuriously zero. The Lipps & Hemler (1982) and Wilhelmson & Ogura (1972) anelastic sets also give significant errors for the deepest internal modes.

It is difficult to make quantitative statements about how accurately the different equation sets represent the partitioning of mode energy among the velocity and thermodynamic variables, because the conserved energy-like quantity (when it exists) is itself defined differently for the different equation sets. But it is clear from (5.20) - (5.22) and (5.27) - (5.28) that the partitioning of energy will be affected in some way, for all internal modes, by making any of the approximations considered. If the amplitude π_0 of the π component of the normal-mode state vector is considered given, then by examining the ratio of the right-hand-side of (5.28) for anelastic equation sets to that of the fully-compressible equations, it is however possible to show that the energy redistribution of the internal Rossby modes (Table 3) can be significant. For example, using the appropriate values of Γ defined by (5.20), the Lipps & Hemler (1982) set changes the thermobaric energy, when $z_T = 5H \approx 40$ km, by more than 20%. The ratio increases (decreases) with vertical wavelength (wavenumber). In general, the Lipps & Hemler (1982) set has larger distortion than the Wilhelmson & Ogura (1972) one due to the larger change induced in Γ.

The analysis has been performed on an f-plane and consequently, as noted in Section 5, the Rossby modes (both external and internal) are degenerate, i.e. $\sigma = 0$. However, conclusions drawn from the above analysis also hold for the non-degenerate case for which f varies with latitude. The argument is as follows. For fully varying f, Daley (1988) and Thuburn et al. (2002*a*) noted that for the *isothermal* basic-state atmosphere and the shallow-atmosphere equations assumed here, the eigen problem for the vertical structure can be solved independently of that for horizontal structure. Furthermore, the vertical structure is independent of the horizontal variation of f. This means that if the analysis were to be redone in spherical geometry, then the same vertical structure as found herein would result with the full variation of f manifesting itself in the horizontal structure. And the Rossby modes would no longer be degenerate.

7 CONCLUSIONS

The fully-compressible unforced inviscid (Euler) equations have been examined using normal mode analysis. This provides an objective framework for assessing the validity of these approximations for various atmospheric modelling applications. The discussion of the previous section leads to the following conclusions:

- The Lamb (external acoustic) modes are filtered out by all approximate sets except the hydrostatic equations which retain them undistorted.
- For external Rossby modes, the anelastic Wilhelmson & Ogura (1972) set is satisfactory in terms of absence of distortion, but energy is not conserved. However, the energy-conserving anelastic Lipps & Hemler (1982) set distorts the height scale. The Boussinesq set both distorts the vertical structure and redistributes energy. The hydrostatic and pseudo-incompressible sets correctly handle these.
- For internal Rossby modes, the two anelastic and the Boussinesq sets significantly misrepresent them at wavelengths typically encountered in atmospheric models; the hydrostatic and pseudo-incompressible sets are the only ones that do not.
- For internal gravity modes, all the approximate sets, apart from the hydrostatic one, significantly mishandle the deep vertical modes at large horizontal scale. However, as the horizontal scale is reduced, the situation is reversed: the hydrostatic equations then misrepresent the deep vertical modes, whereas the other approximate sets handle them quite well, particularly the shallower vertical modes.

For numerical weather prediction and climate simulation, it is essential to properly represent Rossby modes, both external and internal since they carry significant energy. Thus for multiscale applications it is essential that the chosen equation set correctly represents these modes. Of the approximate equation sets considered above, only the hydrostatic and pseudo-incompressible ones do this, although the latter set also filters out the Lamb modes. The Wilhelmson & Ogura (1972) anelastic equations distort the vertical structure of internal Rossby modes, whilst the Lipps & Hemler (1982) anelastic and Boussinesq sets distort both internal and external Rossby modes. Thus neither of these anelastic sets nor the Boussinesq equations are recommended for numerical weather prediction, or for climate simulation, at any scale, but are suitable for process studies with shallow vertical scales since even if Rossby modes are significant, their distortion is small. The pseudo-incompressible set appears to be viable for numerical weather prediction, but only at short horizontal scales;

at large horizontal scales the frequencies of deep gravity modes are distorted. In contrast, the hydrostatic primitive equations perform well at large horizontal scale but, because they neglect the vertical acceleration term Dw/Dt, they are unsuitable for mesoscale flows where this neglected acceleration is important.

Anelastic equation sets are the principal basis of many theoretical and modelling studies of small-scale dynamics, for which they play an analogous role to that of the hydrostatic primitive equations for planetary-scale dynamics. With the advent of global anelastic models (Smolarkiewicz et al. 2001), albeit for research purposes, it is natural to ask the question "are anelastic equation sets suitable for use in operational NWP and climate models?". If the option of nesting models, and possibly employing a different equation set, is admitted, then anelastic models would clearly have a useful role to play provided the Rossby modes are not a fundamental part of the solution. Such a system might consist of a small-scale anelastic model embedded either directly within a large-scale hydrostatic primitive equations model or within an intermediate nonhydrostatic elastic model. However, due to the complexity of such a system and the associated overheads of maintenance, and also the problems that can arise between models employing different equation sets, an arguably more attractive approach for multiscale models is to have one model, and therefore one architecture, which can be successfully run at a range of scales from planetary to mesoscale. This study suggests that the fully-compressible equation set is the only viable candidate. Since then the acoustic oscillations are not filtered from the governing equations, they have to be dealt with numerically. For efficiency reasons short timesteps would normally be ruled out, so implicit or semi-implicit timestepping (e.g. Tanguay, Robert & Laprise (1990)) is required. The global nonhydrostatic fully-compressible models of Cullen, Davies, Mawson, James & Coulter (1997), Qian, Semazzi & Scroggs (1998) and Yeh, Côté, Gravel, Méthot, Patoine, Roch & Staniforth (2002) demonstrate the viability of this approach. [Note however that stability with implicit and semi-implicit timestepping *with long timesteps* is achieved by spuriously retarding the fast-propagating modes responsible for the timestep limitations of explicit schemes. Should fast-propagating oscillations carry non-negligible energy for a given application, then the timestep would have to be shortened in order to properly represent the physics, and the timestep advantage over an explicit treatment would be lost.]

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